Invariants of knots, embeddings and immersions via contact geometry

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1 Associating symplectic objects to smooth ones

1.1 Introduction

Let M be a smooth closed manifold of dimension m. Then its cotangent bundle T^*M has a natural exact symplectic structure $\omega = d\lambda$ coming from the Liouville 1-form λ on T^*M. Recall that λ is defined by the requirement that, for any 1-form α on M (considered as a map $M \to T^*M$) the pullback of λ by α is equal to α :

$$\alpha^*\lambda = \alpha.$$

Explicitly, for $v \in T^*M$, $w \in T_u(T^*M)$, writing $\pi : T^*M \to M$ for the base projection,

$$\lambda_u(w) = u(\pi_*(w)).$$

If we are given local coordinates $q_1, \dots, q_m$ on an open set U in M, then $q_1, \dots, q_m, p_1, \dots, p_m$ are coordinates on $T^*U \subset T^*M$ (where $\alpha \in T^*_q M$ writes $\alpha = \sum_{i=1}^{m} p_i dq_i$). The expression for λ becomes (where by the usual abuse of notation dq_i really means $\pi^* dq_i$, noting that λ is a map $T^*M \to T^*(T^*M)$):

$$\lambda = \sum_{i=1}^{m} p_i dq_i.$$

And we get the standard expression for the symplectic form :

$$\omega = \sum_{i=1}^{m} dp_i \wedge dq_i.$$

Fact 1. If M is diffeomorphic to M', then T^*M is symplectomorphic to T^*M'.

Proof. Let $\phi : M \simeq M'$. Then $\phi_* : T^*M \to T^*M'$ is a diffeomorphism. Now by naturality of the Liouville 1-form (*i.e.* the assignment $M \mapsto \lambda_M$ is functorial), the pullback of $\lambda_{M'}$ by ϕ_* has to be λ_M. As a consequence $(\phi_*)^*(d\lambda_{M'}) = d\lambda_M$. □

*Everything here (content and figures) was stolen from [EE04], [Ng05], [Ng12] and [EENS11].

We conclude that **the symplectomorphism class of T^*M is an invariant of the diffeomorphism class of M**. Is this invariant efficient ?

It turns out the symplectic manifold T^*M recovers homotopical data about M :

Theorem 1.1. *(Viterbo, Salamon-Weber, Abbondandolo-Schwarz). The Hamiltonian Floer homology of T^*M is isomorphic to the singular homology of the free loop space of M.*

It also seems the symplectic structure can distinguish different smooth structures :

Theorem 1.2. *(Abouzaid). If Σ is an exotic (4k+1)-sphere that does not bound a parallelizable manifold, then $T^*\Sigma$ is not symplectomorphic to T^*S^{4k+1} (for the standard sphere).*

Therefore we can wonder : Is the smooth type (*i.e.* up to diffeomorphism) of M determined by the symplectic type of T^*M (*i.e.* up to symplectomorphism) ?
This question is still open (at least for closed manifolds : by the work of Knapp different 4-dimensional homeomorphic open manifolds have symplectomorphic cotangent bundles [Kna12] - thus any exotic $\mathbb{R}^4$ together with standard $\mathbb{R}^4$ are a counterexample).

In this talk, we will present the relative version of this symplectic approach : its application to the classification of types (*i.e.* up to isotopy or regular homotopy) of (immersed) submanifolds.

1.2 The conormal construction

1.2.1 Lagrangian version

Let $N \subset M$ be a n-dimensional (potentially immersed) closed submanifold of M. To it, we can associate a subspace of T^*M : the conormal bundle of N denoted by $\mathcal{L}_N$.

$$\mathcal{L}_N = \{u \in T^*M|_N,\ u(v) = 0 \ \ \forall v \in TN\} \subset T^*M$$

$\mathcal{L}_N$ is the set of covectors based at points of N whose kernel contains the tangent space to N. If N is embedded, $\mathcal{L}_N$ is a vector subbundle of rank $m-n$ of $T^*M|_N$ and as such is an embedded m-submanifold of T^*M. In fact it is exact Lagrangian. Indeed let $u \in \mathcal{L}_N$ and $w \in T_u\mathcal{L}_N \subset T_u(T^*M|_N)$. Note that the base projection π sends $T^*M|_N$ to N, so that $\pi_*(w) \in T_{\pi(u)}N$ is a tangent vector to N and

$$\lambda_u(w) = u(\pi_*(w)) = 0$$

by definition of $\mathcal{L}_N$. We get $\omega|_{\mathcal{L}_N} = d\lambda|_{\mathcal{L}_N} = 0$ and $\lambda|_{\mathcal{L}_N} = 0$ thus exact.
Now via this construction, if we isotope N through smooth embeddings in M, $\mathcal{L}_N$ will be isotoped through exact Lagrangians. So **the hamiltonian isotopy class of $\mathcal{L}_N \subset T^*M$ is an invariant of the smooth isotopy class of $N \subset M$.**

1.2.2 Legendrian version

If we are given a Riemannian metric g on M, we can simplify $\mathcal{L}_N$ a little and make contact geometry enter the game by losing one dimension and getting rid of noncompactness. Let ST^*M be the spherization of the cotangent bundle of M, *i.e.* tangent covectors of norm 1, also called the cosphere bundle. It is a contact manifold which is the convex boundary of the Liouville domain DT^*M (the disc cotangent bundle *i.e.* covectors of norm ≤ 1). Indeed

$\lambda|_{ST^*M}$ is a contact form [1]. Now, intersect $\mathcal{L}_N$ with the unit cotangent bundle ST^*M of M and you get the unit conormal bundle of $N \subset M$:

$$L_N = \mathcal{L}_N \cap ST^*M = \{u \in T^*M|_N,\ ||u||_g = 1,\ u(v) = 0\ \ \forall v \in TN\}.$$

With the isomorphism $T^*M \overset{g}{\to} TM$, we can furthermore identify L_N with the unit normal bundle of N and so with the boundary of a tubular neighborhood of N in M.
The previous discussion shows that L_N is a $(m-n-1)$-sphere bundle over N, and in fact a closed Legendrian submanifold of ST^*M. We conclude as before that **the Legendrian isotopy class of $L_N \subset ST^*M$ is an invariant of the smooth isotopy class of $N \subset M$.**

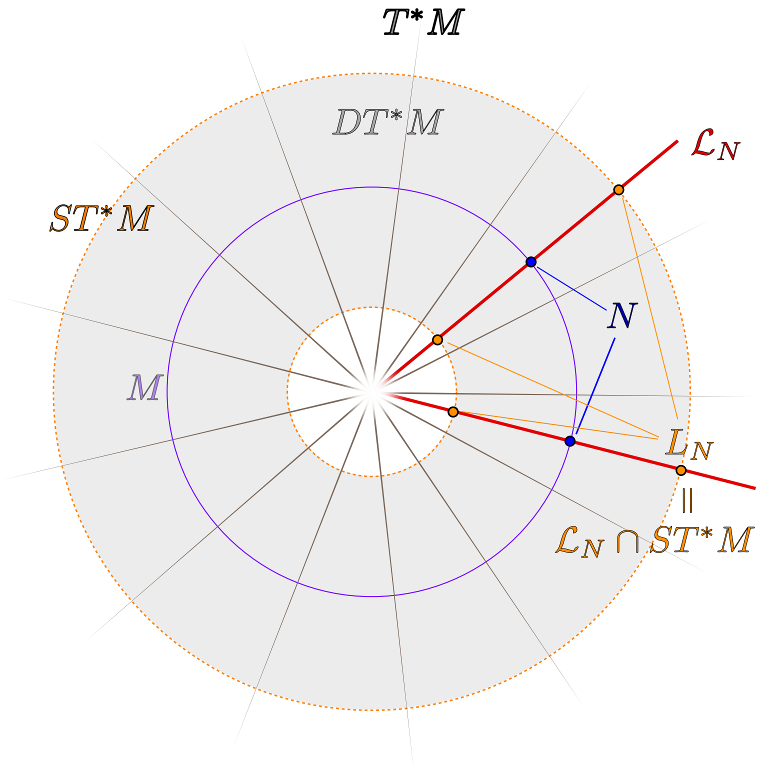

Figure 1: Representation of $N \subset M$, $\mathcal{L}_N \subset T^*M$ and $L_N \subset ST^*M$ in the simplest case $M = S^1$ and N is a pair of points - this can also be considered as schematic of the general case.

Remark 1.3. One could be unconvinced by the last statement because of the necessity of choosing a metric. In fact we can bypass this choice by considering, in place of ST^*M, the Grassmanian bundle of oriented hyperplanes tangent to M. Any tangent hyperplane is determined by a half-line of cotangent vectors of M, so that this bundle is canonically isomorphic to the oriented projectivized cotangent bundle $\mathbb{P}_+T^*M$ of M. As a quotient of T^*M, $\mathbb{P}_+T^*M$ inherits the hyperplane field **ker** λ (because λ commutes with $\mathbb{R}^{>0}$ multiplication in cotangent fibers) and is the natural contact manifold associated to M. Finally in the hyperplane Grassmanian picture, L_N is the set of hyperplanes over points of N which contains the tangent

[1] Let X be the radial vector field in cotangent directions (in local coordinates $X = \sum_{i=1}^{m} p_i \frac{\partial}{\partial p_i}$). It is transverse to ST^*M, and $\lambda = X \lrcorner \omega$. The claim follows.

spaces to N. For any choice of metric, $\mathbb{P}_+T^*M$ can be identified with the unit cotangent bundle ST^*M by a canonical contactomorphism (sending a half-line to its intersection with the unit cosphere). The use of a metric simply allows us to geometrically visualize L_N as the boundary of a tubular neighborhood of N (and clearly see its sphere bundle structure).

Note 1.4. If N is only immersed in M, then L_N could a priori have double points. A simple condition to avoid these is to ask N to be a self-transverse immersion (which is a generic property as $n < m$). In this case, over a double point q of N there is two transverse tangent planes T_qN^1 and T_qN^2, so that there is two corresponding sets of points of L_N over q:

- the oriented tangent hyperplanes of M containing T_qN^1
- the ones containing T_qN^2.

These sets are disjoint because there cannot be a point in both *i.e.* a hyperplane containing T_qN^1 and T_qN^2. Such an hyperplane would contain $T_qN^1 \oplus T_qN^2$, which is T_qM by the transversality assumption, so don't exist. Moreover, if we move N by regular homotopy without self-tangency at double points, L_N remains embedded so that **the Legendrian isotopy class of $L_N \subset ST^*M$ is an invariant of the "regular homotopy without self-tangency" class of the self transverse immersion N in M.**

2 The case of submanifolds of euclidian space

2.1 Generalities

Locally, contact geometry sums up to the study of 1-jet spaces. Recall that if W is a manifold, $J^1W = T^*W \times \mathbb{R}$ is canonically a contact manifold with contact form $\alpha = dz - \lambda_W$ where λ_W is the Liouville 1-form on T^*W lifted trivially in the additional $\mathbb{R}$ factor, and z is this additional coordinate[2]. Let's come back to the setup of the previous section, in the case $M = \mathbb{R}^m$. Then the unit cotangent bundle is trivial :

$$ST^*\mathbb{R}^m = \mathbb{R}^m \times S^{m-1},$$

so that we can see it as S^{m-1}-bundle over $\mathbb{R}^m$, or a $\mathbb{R}^m$-bundle over S^{m-1}. In fact we have better :

Proposition 1. $(ST^*\mathbb{R}^m, \lambda_{\mathbb{R}^m}|_{ST^*\mathbb{R}^m})$ is contactomorphic to $(J^1S^{m-1}, dz - \lambda_{S^{m-1}})$ via the map

$$\begin{array}{ccc} \mathbb{R}^m \times S^{m-1} & \to & J^1(S^{m-1}) \\ (q,p) & \mapsto & \Psi(q,p) = (p, q - \langle q,p\rangle p, \langle q,p\rangle) \end{array}$$

where (q,p) are global coordinates on $\mathbb{R}^m \times S^{m-1}$ (with p of unit norm), and $\langle\cdot,\cdot\rangle$ is the canonical scalar product on $\mathbb{R}^m$, so that $\langle q,p\rangle$ is the part of the vector q orthogonal to the sphere S^{m-1} at p, and $q - \langle q,p\rangle p$ is the part of q tangent to the sphere at p [3].

[2] J^1W is the contactization of the exact symplectic manifold T^*W.

[3] This seems to implicitly assumes a choice of linear isomorphisms between T_pS^{m-1} and $\mathbb{R}^{m-1}$ for each p. This is impossible, except for $m = 2, 4, 8$, because spheres are generally not parallelizable. However the global map is well defined : if we write $\nu(S^{m-1})$ for the normal bundle of the standard sphere in $\mathbb{R}^m$ which is a trivial line bundle, then $TS^{m-1} \oplus \nu(S^{m-1})$ is trivialisable as it is the restriction of the (trivial) tangent bundle of $\mathbb{R}^m$ restricted to S^{m-1} (spheres are *stably* parallelizable). Identifying cotangent and tangent bundles via the canonical metric on $\mathbb{R}^m$, this is exactly what the map Ψ is about (seeing $\mathbb{R}^m \times S^{m-1}$ as $T\mathbb{R}^m|_{S^{m-1}}$.

Proof. Ψ is a bundle map covering the identity of S^{m-1} and is clearly a linear isomorphism on each fiber $p =$ constant. So Ψ it a diffeomorphism. Let's show it is a contactomorphism for the respective canonical contact structures : in coordinates (x, y, z) on J^1S^{m-1} where $x \in \mathbb{R}^m$ unit (assuming the standard embedding of the sphere in euclidian space [4]) and $y \in T_xS^{m-1} \subset \mathbb{R}^m$ so that $y + zx$ is a vector in $\mathbb{R}^m$, we have $\alpha = dz - \lambda_{S^{m-1}} = dz - \sum_{i=1}^{m} y_i dx_i$. So that $\Psi^*\alpha = d(\langle q, p\rangle) - \sum_{i=1}^{m}(q_i - \langle q, p\rangle p_i)dp_i = d(\sum_{i=1}^{m} p_i q_i) - \sum_{i=1}^{m} q_i dp_i$ (because p is on the unit sphere $\sum_{i=1}^{m} p_i^2 = 1$ so that $\sum_{i=1}^{m} p_i dp_i = 0$) and finally $\Psi^*\alpha = \sum_{i=1}^{m} p_i dq_i = \lambda_{\mathbb{R}^m}|_{ST^*\mathbb{R}^m}$. □

The advantage of this point of view are double. In 1-jet spaces there are two special projections : the Lagrangian projection $\pi_{\mathbb{C}} : J^1W \to T^*W$ which forgets the z coordinate and the front projection $\pi_F : J^1W \to W \times \mathbb{R}$ which deletes cotangent directions. We can recover a Legendrian $L \subset J^1W$ from either projection (completely from $\pi_F(L)$ and only up to z-translation from $\pi_{\mathbb{C}}(L)$), and so can diminish greatly the dimension of our drawings (from $2\,\mathbf{dim}\,W + 1$ to $\mathbf{dim}\,W + 1$). Besides, Legendrian contact homology in 1-jet spaces is perfectly well-defined - we will apply it in section 3.

2.2 Plane curves

The simplest non-trivial examples are (connected) plane curves *i.e.* (self-transverse) immersions of $N = \Sigma^1$ in $\mathbb{R}^2$. In this case the unit conormal bundle L_N has two connected components.

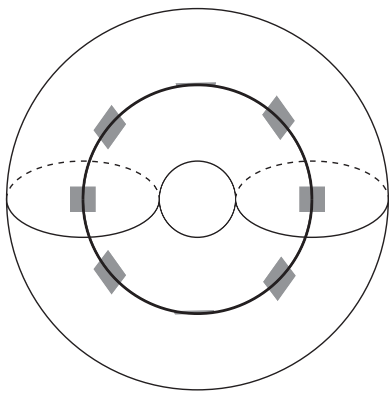

Figure 2: The cosphere bundle of $\mathbb{R}^2$ ($ST^*\mathbb{R}^2$) represented as the filled torus $\mathbb{R}^2 \times S^1$, with some of its contact planes around the $\{0\} \times S^1$ fiber.

In fact this is the general case of codimension one immersions. Indeed as explained earlier we can identify L_N on embedded charts of N with the boundary of a tubular neighborhood of N. As N is a codimension one hypersurface, it cuts the ambiant space in two, so that L_N has (locally) two connected components. What happen when we glue together all of these parts ?

[4]Recall the Liouville form of S^{m-1} is induced by the Liouville form of $\mathbb{R}^m$: $\lambda_{\mathbb{R}^m} = \sum_{i=1}^{m} y_i dx_i$.

Proposition 2.1. *If $n = m - 1$, L_N is a two-fold cover of N which happens to be exactly the orientation cover of N.*

Proof. In the hyperplane Grassmannian picture, remember that L_N is the set of oriented tangent hyperplanes containing TN. But N is an hypersurface so that, if $q \in N$, T_qN is an hyperplane, and so there are only two oriented hyperplanes containing it : $+T_qN$ and $-T_qN$. Each point can be seen as a choice of orientation on T_qN. □

Assuming N is connected, two things can happen :

- either N is not orientable and the orientation cover L_N is connected.
- or N is orientable and L_N has two connected components L_N^+ and L_N^- diffeomorphic to N corresponding to the two possible orientations of N. Each component can be recovered from the other by the antipodal map on S^{m-1} followed by changes of sign on both $\mathbb{R}^{m-1}$ and $\mathbb{R}$ factors (in the $J^1(S^{m-1})$ picture).

Note that in the second case, if we are given an orientation on N, this distinguishes one of the components (denoted L_N^+ above). Moreover, L_N^+ is oriented by this choice. Dropping one component has one more advantage : N can now have inverse self-tangencies. Indeed if at a double point q of N, the tangent planes coïncide but their coorientation (coming from N's orientation together with $\mathbb{R}^m$'s) are opposite (what we mean by an *inverse self-tangency*), each will correspond to only one point. Both form a pair of antipods in S^{m-1}, so that L_N^+ has no double point (whereas $L_N = L_N^+ \cup L_N^-$ has a double point over q). However *direct* self-tangency (*i.e.* when coorientations agree) still creates double points of L_N^+[5]. Calling immersions and regular homotopy *safe* if they don't have direct self-tangency, we conclude : **the Legendrian isotopy class of L_N^+ is a safe homotopy class invariant of the safely immersed (oriented) hypersurface N of M.**

Remark 2.2. In higher codimension $n < m - 1$, N connected implies L_N connected. This is clear when investigating the tubular neighborhood of codim > 1 embedding.

Coming back to plane curves, as S^1 is orientable, this explains our first claim. Recall $ST^*\mathbb{R}^2$ is a contact manifold identified with the filled torus (see figure 2), and L_N (and thus L_N^+) is a Legendrian submanifold of this cosphere bundle. That is to a plane curve, we have associated a Legendrian knot in $\mathbb{R}^2 \times S^1$ (called the *conormal knot*, see figure 3).

So any isotopy invariant of L_N^+ is a safe homotopy invariant of N. For instance we have the classical invariants of L_N^+ :

- its smooth knot type $\mathcal{K}(L_N^+)$ (which contains its homology class in $H^1(\mathbb{R}^2 \times S^1) \simeq \mathbb{Z}$ also called its winding number $n(L_N^+)$ around $\{0\} \times S^1$)
- its rotation number $r(L_N^+)$
- its Thurston-Bennequin number $tb(L_N^+)$.

As explained in the preceding subsection, we can see L_N^+ as a Legendrian in $J^1(S^1)$. Then with (q_1, q_2) coordinates on $\mathbb{R}^2$, parametrizing $J^1(S^1)$ by $(\theta, y, z) \in \mathbb{R}/2\pi\mathbb{Z} \times \mathbb{R} \times \mathbb{R}$, we can

[5]That's why they were called *dangerous* by Arnold.

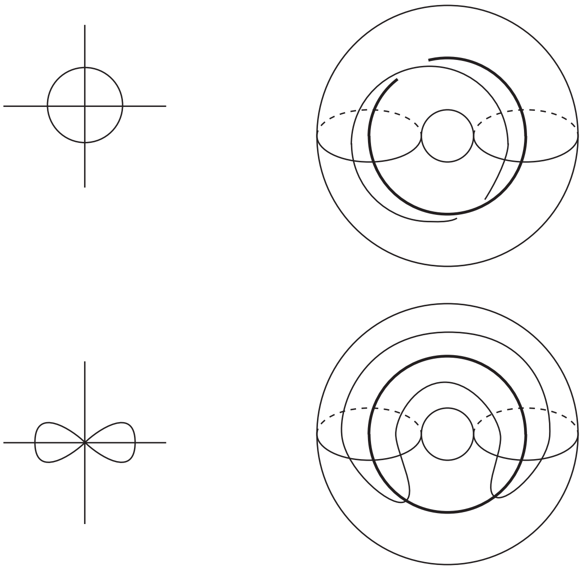

Figure 3: Two oriented planes curves and their Legendrian conormal knots L_N^+ in the filled torus $\mathbb{R}^2 \times S^1$. To get L_N^-, just apply the antipodal map of S^1 to L_N^+.

describe the conormal knot as follows :

$$L_N^+ = \Big\{(\phi + \frac{\pi}{2}, -q_1 \cos\phi - q_2 \sin\phi, -q_1 \sin\phi + q_2 \cos\phi),\ (q_1, q_2) \in N \\ \text{with unit tangent vector } (\cos\phi, \sin\phi)\Big\}.$$

or if we prefer normals to tangents :

$$L_N^+ = \Big\{(\psi, -q_1 \sin\psi + q_2 \cos\psi, q_1 \cos\psi + q_2 \sin\psi),\ (q_1, q_2) \in N \\ \text{with unit normal vector } (\cos\psi, \sin\psi)\Big\}.$$

The front projection is obtained by dropping the middle coordinate y. It is an usual front in $\mathbb{R}/2\pi\mathbb{Z} \times \mathbb{R}$ *i.e.* a singular curve without vertical tangent, and which generically only has transverse double points and semicubical cusps. Cusps of $\pi_F(L_N^+)$ corresponds to inflection points of N.

Figure 4: The same planes curves with their respective Legendrian fronts in $\mathbb{R}/2\pi\mathbb{Z} \times \mathbb{R}$ (coming from the projection $J^1S^1 \to S^1 \times \mathbb{R}$). The first one has index $n = +1$ (with the canonical orientation of the plane), rotation number $r = 0$, and $tb = 0$, whereas the second has $n = 0$, $r = 0$, and $tb = -1$

Computing the classical invariants is now easy (see the examples on figure 4) :

- Round up the cusps and resolve crossings (higher slope behind) to get a smooth knot type. The winding number (index) is just the degree of the map $S^1 \to S^1$ obtained by inclusion of N in $J^1(S^1)$ followed by the projection to the base.

- $r(L_N^+)$ is half the difference between the number of down cusps and the number of up cusps. It turns out to be always zero : there are as many left cusps as right cusps (because $\pi_F(L_N^+)$ is a closed curve without vertical tangent). And left cusps are always down, whereas right cusps are always up (check what happens in a neighborhood of an inflection point of N).

- $tb(L_N^+)$ is the difference between the number of right handed crossings and the number of left handed crossings **minus** half the number of cusps. This is equal to $J^+(L_N^+) + n(L_N^+)^2 + 1$ where J^+ is Arnold framing invariant of plane curves.

In conclusion classical invariants of L_N^+ sums up to the framed knot type of L_N. But this is already a strong invariant of plane curves as can be seen on these examples (figures 4 and 5).

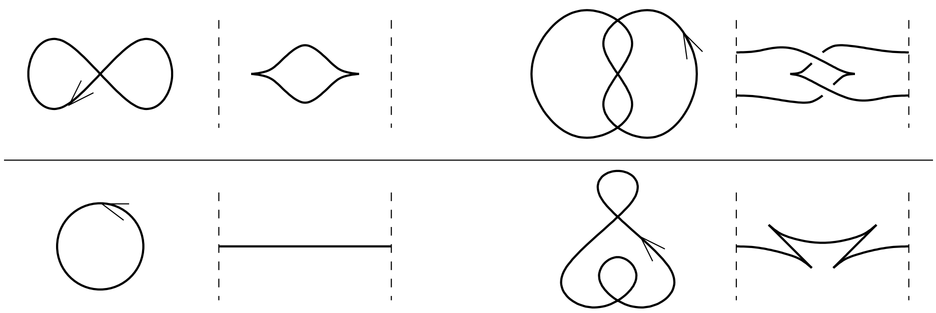

Figure 5: Pairs of plane curves where left and right are distinguished in each line thanks to classical invariants of the conormal knot. Thus any regular homotopy between left and right has to contain a dangerous tangency.

2.3 Surfaces

Consider the case $n = 2 = m - 1$ and more precisely the example of $N = S^2$ safely immersed and oriented. L_N^+ is then a Legendrian sphere in $J^1(S^2)$

Its classical invariants are :

- The safe regular homotopy class of $S^2 \to J^1(S^2)$. As the target is of dimension $2 \times 2 + 1$, by Whitney theorem(s), the only obstruction for two maps $S^2 \to J^1(S^2)$ to be in the same class is of homotopic nature, *i.e.* in $\pi_2(J^1(S^2)) \simeq \pi_2(S^2) \simeq \mathbb{Z}$. This invariant (which is simply the homology class represented by L_K) is determined by the degree of the map $S^2 \to S^2$ obtained by inclusion of L_N^+ followed by projection to the base.

- Its rotation class $r(L_N^+)$ which can be seen as a homotopy class of maps $S^2 \to U(2)$. As $\pi_2(U(2)) = 0$, it it trivial.

- Its Thurston-Bennequin number. It is more or less the linking number of L_N^+ and L_N^-. This gives a first-order invariant of immersions of surfaces into $\mathbb{R}^3$ (an other one is given by the number of pairs of triple points, see [Gor97] for full details).

An well-known example is the standard embedding of the sphere with different coorientations : a path from one to the other is called a *sphere eversion.* Their conormal lifts represent the same homology class but are distinguished by their *tb* so that any sphere eversion must contain dangerous self-tangencies. A more sophisticated example is given on figure 6, we will distinguish them in the next section.

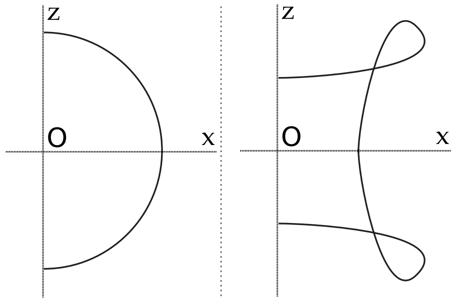

Figure 6: Two immersions of the sphere as surfaces of revolution. They are obtained by spinning each arc (in the plane (xOz)) on the figure around the vertical axis (Oz) joining their ends.

2.4 Knots

Passing to the codimension two case, the simplest example are links. Consider $N = K = S^1$ an oriented embedded knot in $M = \mathbb{R}^3$ We get a Legendrian surface L_K in $J^1(S^2)$. Remembering L_K identifies with the boundary of tubular neighborhood of K, it is clear that L_K is a torus, which we will call (by chance) the *conormal torus* of K. Its classical invariants are :

- The homotopy type of the map $T^2 \to J^1(S^2) \sim S^2$, *i.e.* an element of the second cohomotopy set of the torus. By a theorem of Hopf (as T^2 is a 2-dimensional CW-complex) this set is isomorphic to $H^2(T^2)$. It is determined by the degree of the map $T^2 \to S^2$ obtained by inclusion of L_K followed by projection to the base (it corresponds to the homology class represented by L_K). It is equal to the normal degree of the torus boundary of a small tubular neighborhood of K (by identification of the cotangent and tangent bundle of $\mathbb{R}^3$) which is $\chi(T^2) = 0$ by the Gauss-Bonnet theorem.
- Its rotation class $r(L_K)$ seen as an homotopy class of map $T^2 \to U(2)$. It can be shown to be the null map.
- Its Thurston-Bennequin number. For even-dimensional null-homologous Legendrians, it is always equal to the Euler characteristic, here zero.

In the end, there are no classical invariants for conormal tori. But, we must not lose faith, and proceed to the next best idea :

3 Applying Legendrian contact homology

3.1 General case

Consider a Legendrian Λ in the contact manifold J^1W. As you know, Legendrian contact homology needs the definition of a differential graded algebra with some coefficient (the simplest is $\mathbb{Z}/2$, the reasonably elaborate are $\mathbb{Z}[H^2(J^1W, \Lambda)]$) generated by the Reeb chords of

Λ. To be able to speak of these, we need to fix a contact form α, so that Reeb chords are portions of integral lines of the Reeb vector field R_α which begin and end on Λ. For J^1W, $R_\alpha = \frac{\partial}{\partial z}$ and Reeb chords are vertical segments. Then the differential counts holomorphic curves in the symplectization of J^1W which is $\mathbb{R} \times J^1W$ with symplectic form $d(e^t\alpha)$ where t is the new coordinate. Note that in the case of this talk $W = S^{m-1}$, the symplectization is symplectomorphic to $\dot{T}^*\mathbb{R}^m$ *i.e.* $T^*\mathbb{R}^m$ minus the zero section.

In the front projection Reeb chords correspond to vertical segments joining points with parallel tangent planes, whereas in the Lagrangian projection they correspond to double points.

Proposition 3.1. *For our case of a conormal Legendrian L_N associated to a submanifold N of $\mathbb{R}^m$, the chords correspond to oriented binormals lines of N.*

Definition 3.2. An oriented binormal line[6] is a oriented line joining two points q and q' who share an oriented normal vector p, such that this line is directed by this p.

Proof. Indeed two such points corresponds to $(p, q' - \langle q', p\rangle p, \langle q', p\rangle)$ and $(p, q - \langle q, p\rangle p, \langle q, p\rangle)$ for the same p. These are double points of the Lagrangian projection if and only if $q' - \langle q', p\rangle p = q - \langle q, p\rangle p \iff q' - q = \langle q' - q, p\rangle p$ *i.e.* the line joining q and q' is parallel to the common normal p. □

Note 3.3. We just showed that (oriented) binormals (lines) of N correspond to double points of the Lagrangian projection of L_N. In the same way, oriented bitangent lines, *i.e.* lines joining two points who share a oriented tangent vector v (hence sharing a oriented normal vector p), and are directed by this v, correspond to double points of the front projection of L_N (but these are less important for contact homology). Note also that the length (for the canonical Riemmanian metric on $\mathbb{R}^n$) of a binormal segment is exactly the length of the corresponding Reeb chord, namely $|\langle q' - q, p\rangle|$.

Moreover, this computation shows that binormals come in pairs : if change p in $-p$, we get another oriented binormal line. In terms of chords, this mean that to each Reeb chord over p in $J^1(S^{m-1})$ correspond another Reeb chord over $-p$. Nevertheless, for the codimension one case, where we only consider L_N^+, binormals no longer come by two, because only one of $\{p, -p\}$ is a point of L_N^+ as we fixed an orientation (see figure 7). However if we keep L_N^-, then two cases can happen :

- an oriented binormal line joins two points with the same normal p (fixed by orientation). Then this corresponds to a Reeb chord of L_N^+ over p. But there is also an oriented binormal joining the same points considered with opposite orientation (with normal $-p$). This corresponds to a Reeb chord of L_N^- over $-p$.
- an oriented binormal line joins two points with opposite normal p and $-p$. This is not a pure Reeb chord of L_N^+ nor L_N^-, but this corresponds to two mixed Reeb chords of $L_N = L_N^+ \cup L_N^-$. One over p and one over $-p$.

[6]In standard geometry, a binormal line is simply a unoriented line orthogonal to N in two points. But as we are working with the spherical normal bundle (and not the projectivized normal bundle), orientation of lines matter.

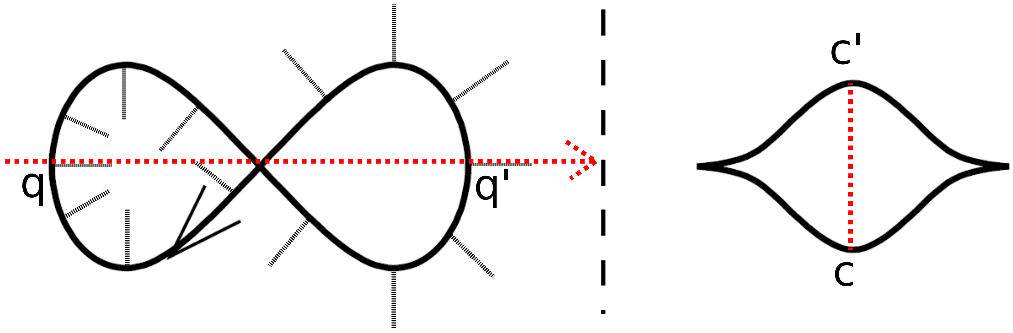

Figure 7: The binormal line (in red dots) on the plane curve N joining q and q' (left), and the corresponding Reeb chord (in red dots) on the front joining the images of these points.

3.2 Plane curves

First-order invariants of plane curves are powerful but not complete. The example on figure 8 of the same plane curve with different orientations cannot be distingued by its framed knot type of its conormal.

Figure 8: Two plane curves and the fronts of their conormal knots. These have the same classical invariants, but are not Legendrian isotopic.

However they are distinguished by their Legendrian contact homology. Indeed taking the satellite of each L_K with the once-positively-stabilized Legendrian unknot, we get the so-called Eliashberg knots $E(2,3)$, and $E(1,4)$ (see figure 9).Their contact DGA are different as their Poincaré polynomials sets are respectively $\{2t + t^{-1}\}$ and $\{t^3 + t + t^{-3}\}$.

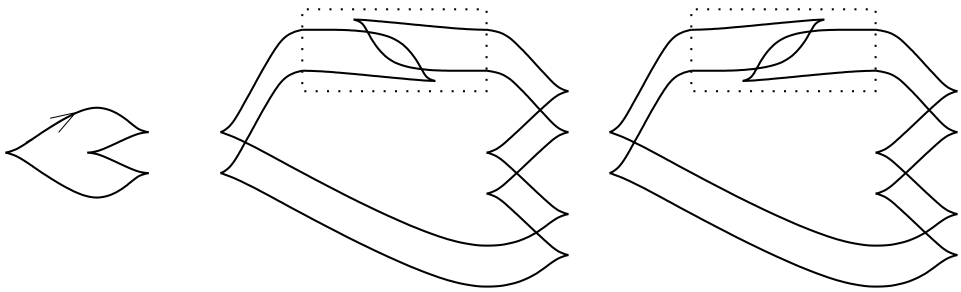

Figure 9: The satellite construction with the stabilized unknot (on the left) on the fronts of figure 8.

3.3 Surfaces

Using the identification of Reeb chords with binormals, we see that the conormal Legendrian associated to the left sphere N_1 on figure 6 has no Reeb chord (for both coorientations). The same can be said of the right sphere N_2, so that both their DGA are $\mathbb{Z}/2$. Let's prove it by showing they have no oriented binormals. As these are surfaces of revolution, the problem reduces to the plane (xOz) (because normal lines to a cross-section stays in the plane of cross-section). In this plane, shown on figure 10, it is clear that there is no oriented binormals (but there are non-oriented binormals corresponding to Reeb chords between L_N^+ and L_N^-).

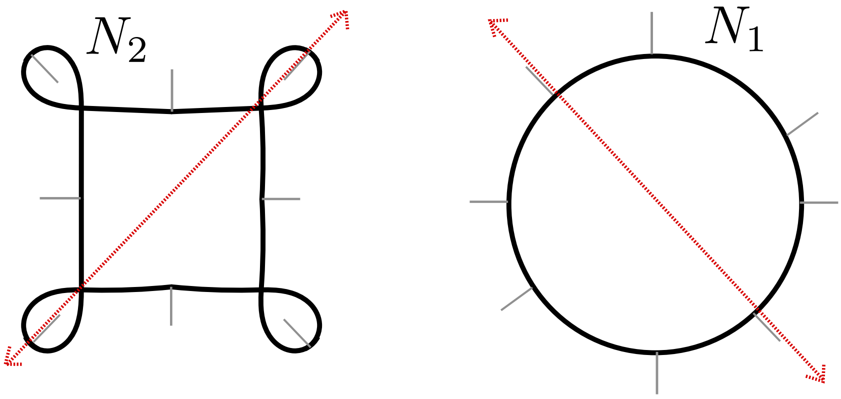

Figure 10: The cross-sections of the spheres N_1 and N_2. On the cross-section of N_1 each radius is an unoriented binormal line, but all pairs of points joined have opposite normals. The same can be said of the 8 unoriented binormal lines we can see on the cross-section of N_2. Thus L_{N_1} and L_{N_2} have no pure Reeb chords, so that $L_{N_1}^+$ and $L_{N_2}^+$ have no Reeb chords.

However, the front of $L_{N_1}^+$ is the zero-section shifted in the vertical (z) direction, so that if we consider the link of $L_{N_1}^+$ and of another zero-section shifted high up L_∞^+ (being the positive conormal lift of a sphere N_∞ of big radius), $L_{N_1}^+ \coprod L_\infty^+$ has a S^2 set of Reeb chords (see figure 11).

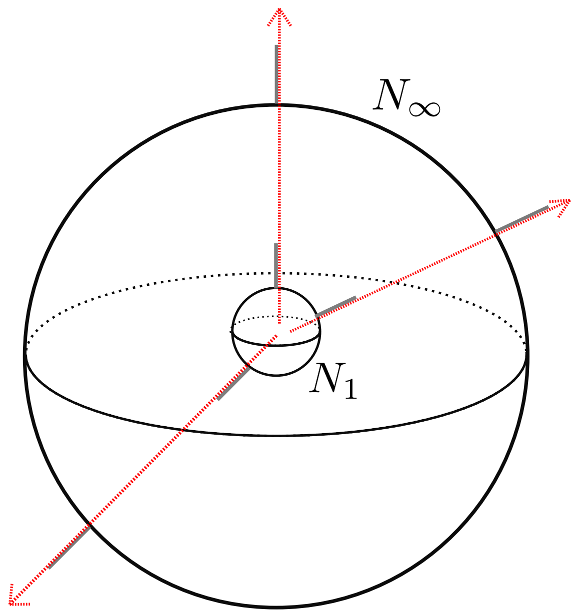

Figure 11: The submanifold $N_1 \coprod N_\infty \subset \mathbb{R}^3$ and some of its oriented binormals. Remember it is the union of two concentric spheres with the same outer coorientation, so that every radius line is an oriented binormal, i.e. we have a S^2 of these.

Perturbing L^+_∞ by a Morse function f, we get only two Reeb chords a_1 and b_1 which are the maximum and minimum of f. By correspondence with the Morse complex of f, $\partial a_1 = 0$ and $\partial b_1 = 0$, so that the Legendrian contact homology of $L^+_{N_1} \coprod L^+_\infty$ is $\mathbb{Z}/2\langle a_1, b_1\rangle$ with $|a_1| = 1+\rho$ and $|b_1| = 0 + \rho$ (for $\rho \in \mathbb{Z}$) as there are no cusps in $\pi_F(L_{N_1})$.

Now, the same can be shown for $L^+_{N_2} \coprod L^+_\infty$ [7] except the two Reeb chords a_2 and b_2 have now a grading difference of 3 (the +2 difference originates from the two loops on the segment between the north and south poles of N_2). Hence $L^+_{N_1}$ and $L^+_{N_2}$ are not Legendrian isotopic, and any path between N_1 and N_2 must have dangerous self-tangencies.

3.4 Knots

The Legendrian contact homology of the conormal torus L_K is called the *knot contact homology* of K. Here we'll sketch its simplest version (*i.e.* with $\mathbb{Z}/2$ coefficients).

3.4.1 The unknot

Let's begin with the standard embedding of the unknot U in the (xOy) plane. We see $S^2 \times \mathbb{R}$ (*i.e.* the codomain of the front projection) as a thickened sphere, and represent only the zero-section. How does the front of L_U look like ?

First, follow what happens to the image of a cotangent circle around a point (figure 12). It corresponds to a shifted (in z direction) great circle of S^2.

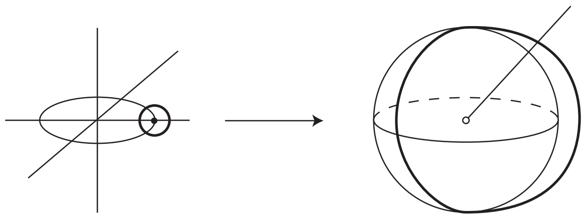

Figure 12: The conormal above a point of the unknot correspond to a great circle above the zero-section $S^2 \times \{0\}$. When the point moves along U, the great circle turn.

Gathering all these cotangent circles, we get the front projection of L_U. We note that it is not front-generic, because $\pi_F(L_U)$ has conical singularities at the poles (whereas it should only have cusps, transverse self-intersections and swallowtails).

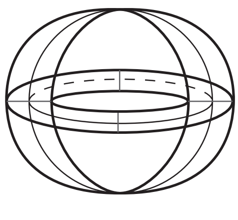

Figure 13: The front of the conormal torus of the unknot U.

[7] Looking at the cross-section picture, there are 3 S^1 of oriented binormal lines and two isolated oriented binormal lines on the axis of revolution joining N_2 and N_∞. But if we perturb N_∞ by an optimal Morse function on the circle, this reduces to the two isolated binormals.

Nevertheless we can now look for Reeb chords. We see a pair of S^1 of these (corresponding to the S^1 family of binormal diameters of U with both orientations) so that it is not chord-generic either. A geometric way of breaking this symmetry is to make the embedding of the unknot elliptic, resulting in only 2 pairs of binormal diameters : one for the minor axis corresponding to minima Reeb chords, and the other for the major axis corresponding to maxima Reeb chords. In fact we can simplify even more by perturbing directly the front of L_U^+ (which will not change Legendrian contact homology) : Choose a optimal Morse function f on S^1 and perturb the front near the equator. The only surviving Reeb chords are the critical points of f namely a minimum c and a maximum e.

Now to compute the gradings and the differential of these two Reeb chords L_U^+, we need to make it front-generic, hence to resolve the conical singularities at the poles. The process of perturbing the front cone is depicted on figure 14. The singularity which projects to a point on the basis S^2 of $J^1(S^2)$, now projects to a curved diamond and its diagonals. The relation to the front is as follows :

- Vertices lift to swallowtails.
- Edges of the external diamond lift to cusp.
- Diagonals of the diamond lift to (transverse) intersection edges.

The detailed structure of these can be reconstructed from figure 15.

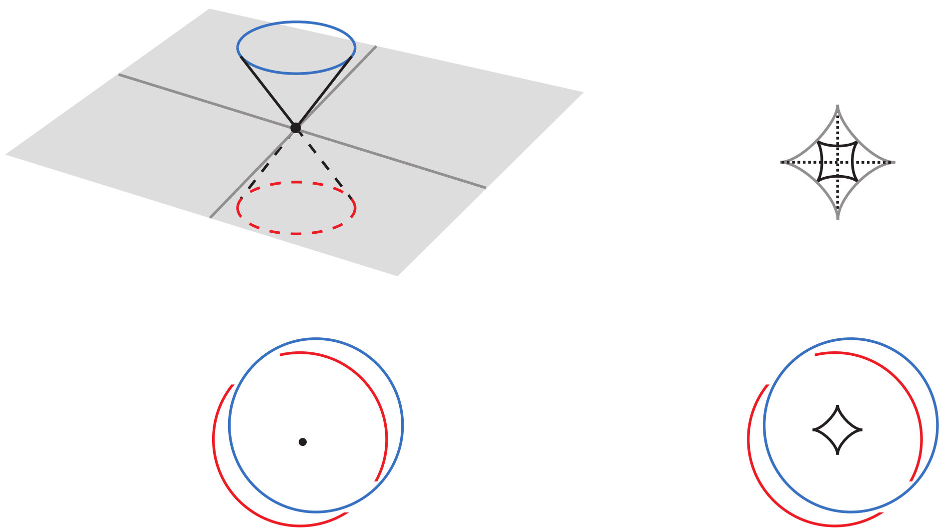

Figure 14: The perturbation of the front cone. On the top left, a small neighborhood of the north pole with the front cone in $S^2 \times \mathbb{R}$. Two circles are distinguished in red and blue. On the bottow left, we have the projection on S^2 of the singularity and the two circles (where the circles are slightly shifted to be both visible). Note the singularity correspond to a degenerate black circle. On the top right, we see the caustic of the projection of the perturbed front cone, with the cusp diamond in grey, intersection edges in dots, and the projection of the black circle which is not degenerate any more. On bottow left, we have the caustic and surrounding circles.

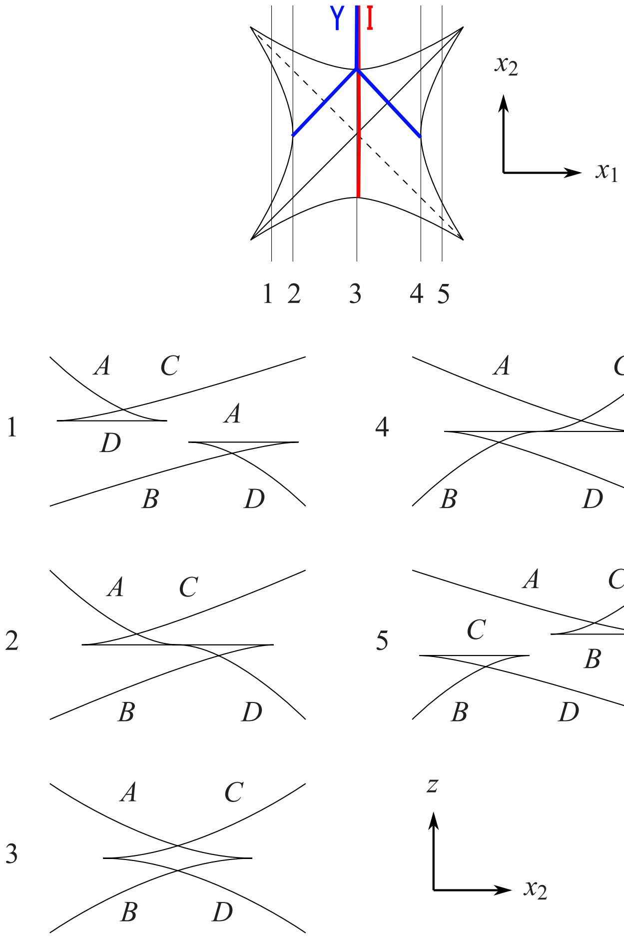

Figure 15: The perturbation of the front cone. Five slices of the front (on top) are shown. This figure is stolen from [Dim11] section 3 where it is explained in detail. See also section 3.1 of [EENS11].

We are now in position to find the gradings of the two Reeb chords. For this, we need to choose capping paths γ_e and γ_c coming from the top of the chord to its bottom. Here we take paths following meridians to the North pole and coming back, see figure 16.

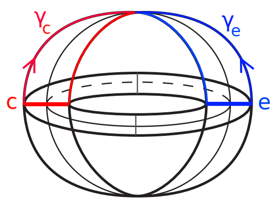

Figure 16: The two Reeb chords and their capping paths.

Recall the formula for the grading of Reeb chords (See lemma 2.5 and 3.1 of [EENS11]) :

$$|c| = \mathbf{cz}(c) - 1 = \mathbf{Ind}_c(f^+ - f^-) + m(\gamma_c) - 1 = \mathbf{Ind}_c(f^+ - f^-) + D - U - 1$$

where **cz** is the Conley-Zehnder of chords, $\mathbf{Ind}_c(f^+ - f^-)$ is the Morse index of the difference of two local functions defining sheets joined by c (with f^+ above f^- in z direction), and m is the Maslov index of (oriented) paths on L_U^+, which is computed by the number of down cusps crossed minus the number of up cusps crossed. γ_e crosses only one down cusp, namely the right cusp of slice 3 of figure 15. Similarly for γ_c, so that we get, recalling c is a minimum, e a maximum in equatorial direction, and both are maxima in meridian direction (*i.e.* $\mathbf{Ind}_c(f^+ - f^-) = 1$ and $\mathbf{Ind}_e(f^+ - f^- = 2)$:

$$|c| = 1 + 1 - 1 = 1 \quad |e| = 2 + 1 - 1 = 2$$

Moreover to ensure the results don't depend on our choices, we must check that the Maslov class of L_U^+ vanishes. This implies checking that Maslov indices of generators of $H_1(L_U^+)$ are zero. Let's take a representative of the longitude of the torus λ along the equator of the sphere, and of the meridian of the torus μ along the meridian of the sphere, as depicted on figure 17. λ crosses no cusp, so its index is zero. And it is clear from the previous discussion that μ crosses one up cusp and one down cusp.

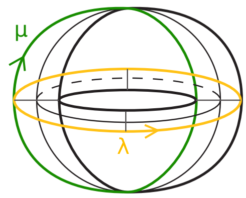

Figure 17: The representatives of the generators of $H_1(L_U^+)$.

We now proceed to the differential. To compute it we will look for rigid Morse flow trees coming out of c and e. As sheets of the front of L_U^+ are maximally far part only at the equator, flow trees stay in a particular hemisphere. Moreover the front is symmetric with respect to north south reversal (we can easily make all pertubations symmetric), so we can represent flow trees on one hemisphere - for example north - and everything will go similarly on the south hemisphere. There are only six rigid trees, two in the north hemisphere, their mirrors in the south, and two in the equator, see figure 18. To explain these, recall that c is a saddle point, the only trees flowing out of c must follow the meridian direction, so they are approching the poles on sheets A and B on slice 3 of figure 15. Now two behaviours are possible :

- I_N (N for North) continue through slice 3 of figure 15 and die on the right cusp AB.
- Y_N split at the left cusp CD in slice 3 (at a trivalent Y vertex). Then one part continue between sheets A and D, to die on the right cusp AD in the middle of slice 2, whereas the other part continue between sheets B and C, to die on the right cusp BC in the middle of slice 4.
- the same can be said of I_S and Y_S respectively.

This amounts to four trees contributing each for 1 in the differential so that $\partial c = 0$ modulo 2.

The chord e is a maximum, so all directions are flowing out of e, but only two are rigid flow trees in the equator, and they have to end on e (they are E_1 and E_2 on 18). These results are in accordance with the expected dimension formula for Morse flow trees :

$$\mathbf{dim}\ \mathcal{M}(c, b_1 b_2 \cdots_n) = |c| - |b_1| \cdots - |b_n| - 1$$

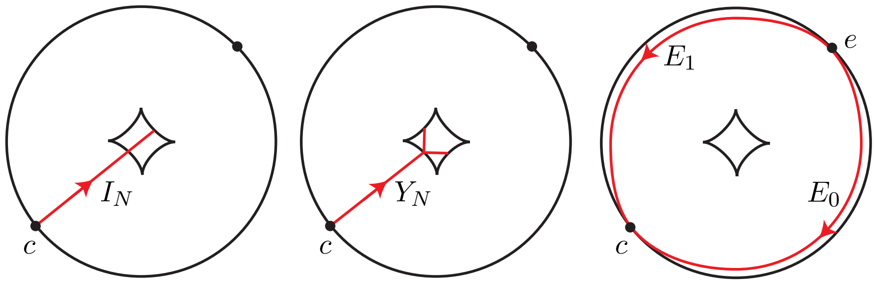

Figure 18: The rigid flow trees coming out of c and e projected on the North hemisphere of the base S^2. The trees I_S and Y_S have the same projection that I_N and Y_N but on the south pole.

We conclude that the Legendrian contact homology of L_U *i.e.* the knot contact homology of the unknot U is $\mathbb{Z}/2\mathbb{Z}$ in degrees 1 and 2, and vanishes in all other degrees.

3.4.2 General knots

For general knots, we have to put them first in a braid representation B, near the unknot. Assume our knot K is entirely contained in a small tubular neighborhood of U. Then apart from the braided part, we get k unknots (where k is the number of strands of B), in a neighborhood of U. So that two kinds of analysis has to be made.

- a local analysis in the tubular neighborhood of U. Looking at binormals between these n strands, perturbing to achieve a minimal number of these, which lead to **small** Reeb chords.
- a global analysis, expanding what we have just done in the preceding section. Between each pair of unknots, there must be two **long** Reeb chords e and c.

And these two must be pieced together. Finally the same kind of two-step reasoning allows one to find the rigid flow trees. The idea is briefly given at the end of [EE04], and full explanations - involving multiscale flow tres - are to be found in [EENS11].

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